Discrete Time Scale Invariant Markov Processes

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Abstract

In this paper we consider a discrete scale invariant Markov process $\{X(t), t \in \mathbf{R}^+\}$ with scale $l > 1$. We consider to have some fix number of observations in every scale, say T, and to get our samples at discrete points α^k, $k \in \mathbf{W}$, where α is obtained by the equality $l = \alpha^T$ and $\mathbf{W} = \{0, 1, \ldots\}$. So we provide a discrete time scale invariant Markov (DT-SIM) process $X(\cdot)$ with parameter space $\{\alpha^k, k \in \mathbf{W}\}$. We present some properties of such a DT-SIM process and we show that the covariance function is characterized by the values of $\{R_j^H(1), R_j^H(0), j = 0, 1, \ldots, T - 1\}$, where $R_j^H(k)$ is the covariance function of jth and $(j + k)$th observations of the process. We also define the corresponding T-dimensional self-similar Markov process and characterize its covariance matrix.

AMS 2000 Subject Classification: 60G18, 60J05, 60G12.

Keywords: Discrete scale invariance; Wide sense Markov; Multi-dimensional self-similar processes.

1 Introduction

The notion of scale invariance or self-similarity is used as a fundamental property to handle and interpret many natural phenomena, like textures in geophysics, turbulence of fluids, data of network traffic and image processing, etc [1]. The idea is that a function is scale invariant if it is identical to any of its rescaled functions, up to some suitable renormalization of its amplitude.

Discrete scale invariance (DSI), a property which requires invariance by dilation for certain preferred scaling factors, was pioneered by Sornette [10]. A random process has DSI if its finite dimensional probability distributions are globally invariant under the action of scaling operators of a fixed ratio [13]. It is known that DSI leads to log-periodic corrections to scaling. Log-periodic oscillations have been used to predict price trends, turbulent time series and crashes on financial markets [11]. Flandrin et. al. have studied the property of DSI and its relation to cyclostationarity by means of the Lamperti transformation [2], [5]. Discrete time linear systems that possess scale invariance properties even in the presence of continuous dilation were proposed by Zhao and Rao [12]. DSI time series in real life usually have some

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scale which is not necessarily integer. Thus we consider some special sampling scheme to obtain a discrete time DSI process, sampled from a continuous time DSI one.

Markov processes have been the center of extensive research activities and wide sense Markov processes are studied before the general theory. In some texts, these processes are defined in the case of transition probabilities of a Markov process. Various classes of wide sense Markov processes are, like jump processes, diffusion processes and processes with a discrete interference of chance [6]. Measurements on a network have shown that LAN traffic behaves as a self-similar process. Through experiments, it has been shown that certain Markov processes can behaves as self-similar processes over a limited but large time domain [9]. By applying both the Markov property and the self-similarity, we have self-similar Markov processes. These processes are involved in various parts of probability theory, such as branching processes and fragmentation theory.

We consider a DSI process with some scale $l > 1$, and we get our samples at points α^k, where $k \in \mathbf{W}$, $l = \alpha^T$, $\mathbf{W} = \{0, 1, \ldots\}$ and T is the number of samples in each scale. By such sampling we provide a discrete time scale invariant Markov (DT-SIM) process in the wide sense which is a discrete time scale invariant process and is Markov in the wide sense too. We find the properties of such a DT-SIM process and also the covariance function is characterized.

The paper is organized as follows. In section 2, we present definitions and some preliminary properties of Markov processes in the wide sense, DSI and self-similar processes by special operators. We define discrete time self-similar and discrete time scale invariant processes too. In section 3 we obtain a closed formula for the covariance function of the DT-SIM process. We also define and characterize the covariance matrix of corresponding T-dimensional self-similar Markov process in section 4.

2   Theoretical framework

In this section, by using renormalized dilation operator, we introduce self-similar and wide sense self-similar processes and define the concepts of discrete time self-similar and discrete time scale invariant processes. We also present some characterizations of Markov processes in the wide sense.

2.1   Stationary and self-similar processes

Definition 2.1 *Given $\tau \in \mathbf{R}$, the shift operator $\mathcal{S}_\tau$ operates on process $\{Y(t), t \in \mathbf{R}\}$ according to*

$$(\mathcal{S}_\tau Y)(t) := Y(t + \tau). \tag{2.1}$$

A process $\{Y(t), t \in \mathbf{R}\}$ is said to be stationary, if for any $t, \tau \in \mathbf{R}$

$$\{(\mathcal{S}_\tau Y)(t)\} \stackrel{d}{=} \{Y(t)\} \tag{2.2}$$

where $\stackrel{d}{=}$ is the equality of all finite-dimensional distributions.

If (2.2) holds for some $\tau \in \mathbf{R}$, the process is said to be periodically correlated. The smallest of such τ is called period of the process.

Definition 2.2 *Given some numbers $H > 0$ and $\lambda > 0$, the renormalized dilation operator $\mathcal{D}_{H,\lambda}$ operates on process $\{X(t), t \in \mathbf{R}^+\}$ according to*

$$(\mathcal{D}_{H,\lambda}X)(t) := \lambda^{-H}X(\lambda t). \tag{2.3}$$

A process $\{X(t), t \in \mathbf{R}^+\}$ is said to be self-similar of index H, if for any $\lambda > 0$

$$\{(\mathcal{D}_{H,\lambda}X)(t)\} \stackrel{d}{=} \{X(t)\}. \tag{2.4}$$

The process is said to be DSI of index H and scaling factor $\lambda_0 > 0$ or (H, λ_0)-DSI, if (2.4) holds for $\lambda = \lambda_0$.

Definition 2.3 *A process $\{X(k), k \in \check{\mathbf{T}}\}$ is called discrete time self-similar process with parameter space $\check{T}$, any subset of distinct points of real line, if for any $k_1, k_2 \in \check{T}$*

$$\{X(k_2)\} \stackrel{d}{=} (\frac{k_2}{k_1})^H \{X(k_1)\}. \tag{2.5}$$

The process $X(\cdot)$ is called discrete time scale invariance with scale $L > 0$ and parameter space $\check{T}$, if for any $k_1, k_2 = Lk_1 \in \check{T}$, (2.5) holds.

Remark 2.1 *If the process $\{X(t), t \in \mathbf{R}^+\}$ is DSI with scale $l > 1$, then for any fixed $s > 0$, $X(\cdot)$ with parameter space $\check{T} = \{l^k s; k \in \mathbf{Z}\}$ is a discrete time self-similar process. If the process $\{X(t), t \in \mathbf{R}^+\}$ is DSI with scale $l = \alpha^T$, for some $T \in \mathbf{N}$ and $\alpha > 1$, then for any fixed $s > 0$, $X(\cdot)$ with parameter space $\check{T} = \{\alpha^k s; k \in \mathbf{Z}\}$ is a discrete time scale invariant process, with the same scale $l = \alpha^T$.*

Remark 2.2 *If the process $\{X(t), t \in \mathbf{R}^+\}$ is DSI with scale $l = \alpha^T$ for fixed $T \in \mathbf{N}$ and $\alpha > 1$, then by sampling of the process at points $\alpha^k, k \in \mathbf{W}$ where $\mathbf{W} = \{0, 1, \ldots\}$, we have $X(\cdot)$ as a discrete time scale invariant process with parameter space $\check{T} = \{\alpha^k; k \in \mathbf{W}\}$ with scale $l = \alpha^T$. If we consider sampling of $X(\cdot)$ at points $\alpha^{nT+k}, n \in \mathbf{W}$, for fixed $k = 0, 1, \ldots, T-1$, then $X(\cdot)$ is a discrete time self-similar process with parameter space $\check{T} = \{\alpha^{nT+k}; n \in \mathbf{W}\}$.*

Definition 2.4 *A random process $\{X(t), t \in \mathbf{R}^+\}$ is said to be wide sense self-similar with index H, for some $H > 0$ if the following properties are satisfied for each $a > 0$ [8]*

(i) $E[X^2(t)] < \infty$,
(ii) $E[X(at)] = a^H E[X(t)]$,
(iii) $E[X(at_1)X(at_2)] = a^{2H} E[X(t_1)X(t_2)]$.

This process is called wide sense discrete scale invariance of index H and scaling factor $a_0 > 0$, if the above conditions hold for some $a = a_0$.

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Definition 2.5 *A random process $\{X(t), t \in \mathbf{R}^+\}$ is said to be discrete time self-similar (DT-SS) in the wide sense with parameter space $\check{T} = \{\lambda^n s, n \in \mathbf{W}, \lambda > 1\}$ for any fixed $s > 0$ and index $H > 0$, if the followings are satisfied for every $n, n_1, n_2 \in \mathbf{W}$*

(i) $E[X^2(\lambda^n s)] < \infty$,
(ii) $E[X(\lambda^{n+n_1}s)] = \lambda^{nH} E[X(\lambda^{n_1}s)]$,
(iii) $E[X(\lambda^{n+n_1}s)X(\lambda^{n+n_2}s)] = \lambda^{2nH} E[X(\lambda^{n_1}s)X(\lambda^{n_2}s)]$.

If the above conditions hold for fixed n, then the process is called discrete time scale invariance (DT-SI) in the wide sense with scale $l = \lambda^n$.

Through this paper we are dealt with wide sense self-similar and wide sense scale invariant process, and for simplicity we omit the term "in the wide sense" hereafter.

2.2 Discrete time Markov processes

Let $\{X(n), n \in \mathbf{Z}\}$ be a second order process of centered random variables, $E[X(n)] = 0$ and $E[|X(n)|^2] < \infty$, $n \in \mathbf{Z}$. Following Doob [4], a real valued second order process $\{X(n), n \in \mathbf{Z}\}$ is Markov in the wide sense if, whenever $t_1 < \ldots < t_n$,

$$\hat{E}\big[X(t_n)|X(t_1), \ldots, X(t_{n-1})\big] = \hat{E}\big[X(t_n)|X(t_{n-1})\big]$$

is satisfied with probability 1, where $\hat{E}$ stands for the linear projection (minimum variance estimator). If the process is Gaussian, then $\hat{E}$ is a version of the conditional expectation. The following facts on covariance of Markov processes in the wide sense, are essentially due to Doob [4]. Let

$$\varrho(n_1, n_2) = \begin{cases} R(n_1, n_2)/R(n_1, n_1) & R(n_1, n_1) > 0 \\ 0 & R(n_1, n_1) = 0 \end{cases}$$

where $R(n_1, n_2) := E[X(n_1)X(n_2)]$ is the covariance function of $X(\cdot)$. Then $\{X(n), n \in \mathbf{Z}\}$ is Markov if and only if ϱ satisfies the functional equation

$$\varrho(n_1, n_2) = \varrho(n_1, n)\varrho(n, n_2), \qquad n_1 \leqslant n \leqslant n_2$$

which is the same as

$$R(n_1, n)R(n, n_2) = R(n, n)R(n_1, n_2), \qquad n_1 \leqslant n \leqslant n_2.$$

It follows that

$$R(n_1, n_2) = G(n_1)K(n_2), \qquad n_1 \leqslant n_2$$

for some functions G and K [7].

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Borisov [3] completed the circle even for continuous time processes, namely, let $R(t_1, t_2)$ be some function defined on $\mathcal{T} \times \mathcal{T}$ and suppose that $R(t_1, t_2) \neq 0$ everywhere on $\mathcal{T} \times \mathcal{T}$, where $\mathcal{T}$ is an interval. Then for $R(t_1, t_2)$ to be the covariance function of a Gaussian Markov process with time space $\mathcal{T}$ it is necessary and sufficient that

$$R(t_1, t_2) = G\big(\min(t_1, t_2)\big) K\big(\max(t_1, t_2)\big) \tag{2.6}$$

where G and K are defined uniquely up to a constant multiple and the ratio G/K is a positive nondecreasing function on $\mathcal{T}$.

It should be noted that the Borisov result on Gaussian Markov processes can be easily derived in the discrete case for second order Markov processes in the wide sense, by using theorem 8.1 of Doob [4].

3   Covariance structure

In this section we study the structure of the covariance function of discrete time scale invariant Markov process and denote it by DT-SIM. Let $\{X(t), t \in \mathbf{R}^+\}$ be a mean zero DT-SIM process with scale l. If $l < 1$, we reduce the time scale, so that l be greater than 1 in the new scale. Now we consider to have T samples in each scale. Our sampling scheme is to get samples at points $1, \alpha, \alpha^2, \ldots, \alpha^{T-1}, \alpha^T, \alpha^{T+1}, \ldots$, where α is obtained by equality $l = \alpha^T$. Thus the process under study will be $\{X(\alpha^n), n \in \mathbf{W}\}$ which is DT-SIM with scale $l = \alpha^T$. We show that the covariance function $R_n^H(\cdot)$ of DT-SIM is specified by the values of $\{R_j^H(1), R_j^H(0), j = 0, 1, \ldots, T-1\}$. We present a closed formula for the covariance function of DT-SIM process in theorem 3.1. The covariance function of DT-SIM process under some conditions is characterized in theorem 3.3.

Theorem 3.1 *Let $\{X(\alpha^n), n \in \mathbf{W}\}$ be a* DT-SIM *process with scale $l = \alpha^T$, $\alpha > 1$, $T \in \mathbf{N}$, then covariance function*

$$R_n^H(\tau) \equiv R_X(\alpha^{n+\tau}, \alpha^n) := E[X(\alpha^{n+\tau})X(\alpha^n)] \tag{3.1}$$

where $\tau \in \mathbf{W}$, $n = 0, 1, \ldots, T-1$ and $R_n^H(\tau) \neq 0$ is of the form

$$R_n^H(kT + v) = [\tilde{h}(\alpha^{T-1})]^k \tilde{h}(\alpha^{v+n-1})[\tilde{h}(\alpha^{n-1})]^{-1} R_n^H(0) \tag{3.2}$$

where $k \in \mathbf{W}$, $v = 0, 1, \ldots, T-1$

$$\tilde{h}(\alpha^r) = \prod_{j=0}^{r} h(\alpha^j) = \prod_{j=0}^{r} R_j^H(1)/R_j^H(0), \qquad r \in \mathbf{W} \tag{3.3}$$

and $\tilde{h}(\alpha^{-1}) = 1$.□

As $\{X(\alpha^n), n \in \mathbf{W}\}$ is DT-SI with scale α^T, this theorem fully characterize the covariance function of the DT-SIM process. Before proceeding to the proof of the theorem we present some equalities about the covariance function of the process which is needed in the proof of the theorem.

From the Markov property (2.6), for $\alpha > 1$, $R_n^H(\tau)$ satisfies

$$R_n^H(\tau) = G(\alpha^n)K(\alpha^{n+\tau}) \qquad \tau \in \mathbf{W}. \tag{3.4}$$

By substituting $\tau = 0$ in the above relation we have that

$$R_n^H(0) = G(\alpha^n)K(\alpha^n) \Longrightarrow G(\alpha^n) = \frac{R_n^H(0)}{K(\alpha^n)}.$$

Therefore

$$R_n^H(\tau) = \frac{K(\alpha^{n+\tau})}{K(\alpha^n)}R_n^H(0), \qquad \tau \in \mathbf{W} \tag{3.5}$$

$$\Longrightarrow K(\alpha^{n+\tau}) = \frac{R_n^H(\tau)}{R_n^H(0)}K(\alpha^n).$$

Now it follows that

$$K(\alpha^{n+1}) = \frac{R_n^H(1)}{R_n^H(0)}K(\alpha^n) = \cdots = \frac{R_n^H(1)}{R_n^H(0)}\frac{R_{n-1}^H(1)}{R_{n-1}^H(0)}\cdots\frac{R_0^H(1)}{R_0^H(0)}K(1).$$

Thus

$$K(\alpha^n) = K(1)\prod_{j=0}^{n-1}h(\alpha^j) \tag{3.6}$$

where $h(\alpha^j) = R_j^H(1)/R_j^H(0)$. Hence for $n = 0, 1, \ldots, T-1, \ k \in \mathbf{W}$

$$K(\alpha^{kT+n}) = K(1)\prod_{j=0}^{kT+n-1}h(\alpha^j).$$

As $X(\cdot)$ is DT-SI with parameter space $\{\alpha^k, k \in \mathbf{W}\}$ and scale α^T by (3.1)

$$\frac{R_{T+i}^H(1)}{R_{T+i}^H(0)} = \frac{R_i^H(1)}{R_i^H(0)}, \qquad i \in \mathbf{W}. \tag{3.7}$$

Therefore

$$\prod_{j=0}^{kT+n-1}h(\alpha^j) = \begin{cases} \left[\prod_{j=0}^{T-1}h(\alpha^j)\right]^k \prod_{j=0}^{n-1}h(\alpha^j) & n > 0 \\ \left[\prod_{j=0}^{T-1}h(\alpha^j)\right]^k & n = 0 \end{cases}$$

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Thus using (3.3) and the convention $\tilde{h}(\alpha^{-1}) = 1$ we have

$$\prod_{j=0}^{kT+n-1} h(\alpha^j) = [\tilde{h}(\alpha^{T-1})]^k \tilde{h}(\alpha^{n-1}). \tag{3.8}$$

Consequently for $n = 0, 1, \ldots, T - 1$

$$K(\alpha^{kT+n}) = K(1)\left[\tilde{h}(\alpha^{T-1})\right]^k \tilde{h}(\alpha^{n-1}). \tag{3.9}$$

Proof of theorem 3.1: Let $\tau = kT + v$, then it follows from (3.5) and (3.9) that

$$R_n^H(kT + v) = \frac{K(\alpha^{n+kT+v})}{K(\alpha^n)} R_n^H(0) = \frac{K(1)[\tilde{h}(\alpha^{T-1})]^k \tilde{h}(\alpha^{v+n-1})}{K(1)\tilde{h}(\alpha^{n-1})} R_n^H(0)$$

$$= [\tilde{h}(\alpha^{T-1})]^k \tilde{h}(\alpha^{v+n-1})[\tilde{h}(\alpha^{n-1})]^{-1} R_n^H(0)$$

for $k = 0, 1, \ldots$, $\alpha > 1$ and $n, v = 0, 1, \ldots, T - 1$. $\square$

Using theorem 3.1, one can easily verify that

$$\sum_{\tau=0}^{\infty} |R_n^H(\tau)| = \sum_{k=0}^{\infty} |R_n^H(kT + v)| = \sum_{k=0}^{\infty} |[\tilde{h}(\alpha^{T-1})]^k \tilde{h}(\alpha^{v+n-1})[\tilde{h}(\alpha^{n-1})]^{-1} R_n^H(0)|$$

$$= |\tilde{h}(\alpha^{v+n-1})[\tilde{h}(\alpha^{n-1})]^{-1}| |R_n^H(0)| \sum_{k=0}^{\infty} |\tilde{h}(\alpha^{T-1})|^k$$

Now the process is short memory if $\tilde{h}(\alpha^{T-1}) < 1$, otherwise the process is long memory.

Remark 3.2 It follows from theorem 3.1 and relations (3.3) and (3.7) that $R_n^H(\tau)$, $n = 0, 1, \ldots, T - 1$ is fully specified by the values of $\{R_j^H(1), R_j^H(0), j = 0, 1, \ldots, T - 1\}$.

By the following theorem, the necessary conditions are given to show that, the $2T$ elements $\{R_j^H(1), R_j^H(0), j = 0, 1, \ldots, T - 1\}$ of the function $R_n^H(\tau)$, given by (3.2) characterize the covariance function of the DT-SIM process.

Theorem 3.3 The function $R_n^H(\tau)$ in relation (3.2) characterize the covariance function of a DT-SIM process, if these covariance functions satisfy the following condition

$$\left(R_j^H(1)\right)^2 \leqslant R_j^H(0) R_{j+1}^H(0) \tag{3.10}$$

for $j = 0, 1, \ldots, T - 1$ and $R_T^H(0) = \alpha^{2TH} R_0^H(0)$.

Proof: As by theorem 3.1 we have that the covariance function of the DT-SIM process is of the form (3.2), so for the proof of this theorem it is enough to show that every covariance function of the form (3.2) is the covariance function of a DT-SIM process. For this, we show in (i) that the covariance function $R_n^H(\tau)$ which is defined in (3.1) has scale invariance property with scale $l = \alpha^T$, i.e, satisfies the equality $R_{n+T}^H(kT + v) = l^{2H} R_n^H(kT + v)$. We also prove in (ii) that $R_n^H(\tau)$ is the covariance function of a wide sense Markov process, as it satisfies in the relation (2.6).

(i) According to (3.2) we have that

$$R_{n+T}^H(kT + v) = [\tilde{h}(\alpha^{T-1})]^k \tilde{h}(\alpha^{v+n+T-1})[\tilde{h}(\alpha^{n+T-1})]^{-1} R_{n+T}^H(0)$$

where by (3.3) and (3.7)

$$\frac{\tilde{h}(\alpha^{v+n+T-1})}{\tilde{h}(\alpha^{n+T-1})} = \frac{\prod_{j=0}^{v+n+T-1} \frac{R_j^H(1)}{R_j^H(0)}}{\prod_{j=0}^{n+T-1} \frac{R_j^H(1)}{R_j^H(0)}} = \frac{\prod_{j=0}^{T-1} \frac{R_j^H(1)}{R_j^H(0)} \prod_{j=T}^{v+n+T-1} \frac{R_j^H(1)}{R_j^H(0)}}{\prod_{j=0}^{T-1} \frac{R_j^H(1)}{R_j^H(0)} \prod_{j=T}^{n+T-1} \frac{R_j^H(1)}{R_j^H(0)}}$$

$$= \frac{\prod_{j=0}^{v+n-1} \frac{R_{j+T}^H(1)}{R_{j+T}^H(0)}}{\prod_{j=0}^{n-1} \frac{R_{j+T}^H(1)}{R_{j+T}^H(0)}} = \frac{\prod_{j=0}^{v+n-1} \frac{R_j^H(1)}{R_j^H(0)}}{\prod_{j=0}^{n-1} \frac{R_j^H(1)}{R_j^H(0)}} = \frac{\tilde{h}(\alpha^{v+n-1})}{\tilde{h}(\alpha^{n-1})},$$

and

$$R_{n+T}^H(0) = E[X(\alpha^{n+T})X(\alpha^{n+T})] = \alpha^{2TH} E[X(\alpha^n)X(\alpha^n)] = l^{2H} R_n^H(0).$$

Thus $R_{n+T}^H(kT + v) = l^{2H} R_n^H(kT + v)$.

(ii) By (3.1) and (3.2) we have that

$$R_n^H(kT + v) = R_X(\alpha^{kT+v+n}, \alpha^n) = [\tilde{h}(\alpha^{T-1})]^k \tilde{h}(\alpha^{v+n-1})[\tilde{h}(\alpha^{n-1})]^{-1} R_n^H(0)$$

$$= K(1)[\tilde{h}(\alpha^{T-1})]^k \tilde{h}(\alpha^{v+n-1})[K(1)\tilde{h}(\alpha^{n-1})]^{-1} R_n^H(0).$$

According to (3.8)

$$R_n^H(kT + v) = K(1)\tilde{h}(\alpha^{kT+n+v-1})[K(1)\tilde{h}(\alpha^{n-1})]^{-1} R_n^H(0)$$

$$= K(\alpha^{kT+n+v})[K(\alpha^n)]^{-1} R_n^H(0) = K(\alpha^{kT+n+v})G(\alpha^n)$$

where by (3.4), (3.9)

$$K(\alpha^n) = K(1)\tilde{h}(\alpha^{n-1}), \quad G(\alpha^n) = [K(\alpha^n)]^{-1} R_n^H(0).$$

Thus for $\tau \in \mathbf{W}$, $\alpha > 1$

$$R_n^H(\tau) = R_X(\alpha^{n+\tau}, \alpha^n) = K(\alpha^{n+\tau})G(\alpha^n).$$

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As $\alpha^{n+\tau} > \alpha^n$, by Borisov condition (2.6)

$$R_X(\alpha^{n+\tau}, \alpha^n) = G\big(min(\alpha^{n+\tau}, \alpha^n)\big) K\big(max(\alpha^{n+\tau}, \alpha^n)\big) = G(\alpha^n)K(\alpha^{n+\tau}).$$

Finally we show that G/K is positive and nondecreasing. Note that

$$\frac{G(\alpha^n)}{K(\alpha^n)} = \frac{R_n^H(0)}{K^2(\alpha^n)} > 0. \tag{3.11}$$

Now we prove that $F(\alpha^r) = G(\alpha^r)/K(\alpha^r)$ under condition (3.10) is nondecreasing function. So we show that $F(\alpha^{r+1})/F(\alpha^r) \geqslant 1$. Let $r = kT + n \geqslant 0$ and $0 \leqslant n \leqslant T - 1$, then we consider two cases, $0 \leqslant n \leqslant T - 2$ and $n = T - 1$.

For $0 \leqslant n \leqslant T - 2$, by (3.11) and (3.9) we have that

$$\frac{F(\alpha^{r+1})}{F(\alpha^r)} = \frac{K^2(\alpha^{kT+n})}{K^2(\alpha^{kT+n+1})} \frac{R_{kT+n+1}^H(0)}{R_{kT+n}^H(0)} = \left[\frac{K(1)[\tilde{h}(\alpha^{T-1})]^k \tilde{h}(\alpha^{n-1})}{K(1)[\tilde{h}(\alpha^{T-1})]^k \tilde{h}(\alpha^n)}\right]^2 \frac{R_{n+1}^H(0)}{R_n^H(0)}.$$

As $X(\cdot)$ is DT-SI with parameter space $\{\alpha^k, k \in \mathbf{W}\}$ and scale α^T, so by (3.1)

$$R_{kT+n}^H(0) = R_X(\alpha^{kT+n}, \alpha^{kT+n}) = E[X(\alpha^{kT+n})X(\alpha^{kT+n})]$$

$$= (\alpha^T)^{2kH} E[X(\alpha^n)X(\alpha^n)] = \alpha^{2kTH} R_n^H(0).$$

Therefore

$$\frac{F(\alpha^{r+1})}{F(\alpha^r)} = [\frac{\tilde{h}(\alpha^{n-1})}{\tilde{h}(\alpha^n)}]^2 \frac{R_{n+1}^H(0)}{R_n^H(0)} = [\frac{R_n^H(0)}{R_n^H(1)}]^2 \frac{R_{n+1}^H(0)}{R_n^H(0)} = \frac{R_n^H(0)R_{n+1}^H(0)}{(R_n^H(1))^2}.$$

Under condition (3.10) for $j = n$, $F(\alpha^{r+1})/F(\alpha^r) \geqslant 1$ and $F(\alpha^r)$ is nondecreasing.

For $n = T - 1$ we have

$$\frac{F(\alpha^{r+1})}{F(\alpha^r)} = \frac{K^2(\alpha^{kT+T-1})}{K^2(\alpha^{kT+T})} \frac{R_{kT+T}^H(0)}{R_{kT+T-1}^H(0)} = [\frac{\tilde{h}(\alpha^{T-2})}{\tilde{h}(\alpha^{T-1})}]^2 \frac{R_T^H(0)}{R_{T-1}^H(0)}$$

$$[\frac{R_{T-1}^H(0)}{R_{T-1}^H(1)}]^2 \frac{R_T^H(0)}{R_{T-1}^H(0)} = \frac{R_{T-1}^H(0)R_T^H(0)}{(R_{T-1}^H(1))^2}.$$

By a similar method and under condition (3.10) for $j = T - 1$, $F(\alpha^r)$ is nondecreasing.$\square$

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4 T-dimensional self-similar Markov processes

In this section we assume that $\{X(\alpha^n), n \in \mathbf{W}\}$ is a DT-SIM process with scale $l = \alpha^T$, and characterize the covariance function of the associated T-dimensional discrete time self-similar Markov process in theorem 4.1.

Definition 4.1 *The process* $U(t) = (U^0(ts_0), U^1(ts_1), \ldots, U^{T-1}(ts_{T-1}))$ *for fixed positive* $s_0, s_1,$
$\cdots, s_{T-1}$ *and with parameter space* $\check{T} = \{l^n; n \in \mathbf{W}, l > 1\}$ *is a T-dimensional discrete time self similar if the followings are satisfied*

(a) $\{U^j(t)\}$ *for all* $j = 0, 1, \cdots, T-1$ *is a DT-SS process with parameter space*
$\check{T}^j = \{l^n s_j; n \in \mathbf{W}\}.$

(b) $U^j(\cdot)$ *and* $U^i(\cdot)$ *has self-similar correlation, that is*

$$\mathrm{Cov}\big(U^j(l^{n+n_1}s_j)U^i(l^{n+n_2}s_i)\big) = l^{2nH}\mathrm{Cov}\big(U^j(l^{n_1}s_j)U^i(l^{n_2}s_i)\big)$$

for every $n, n_1, n_2 \in \mathbf{W}, \ i, j = 0, 1, \cdots, T-1.$

Corresponding to the DT-SIM process $\{X(\alpha^n), n \in \mathbf{W}\}$ with scale $l = \alpha^T$, $\alpha > 1$ there exist a T-dimensional discrete time self-similar Markov process $V(t) = \big(V^0(t), V^1(t), \ldots, V^{T-1}(t)\big)$ with parameter space $\check{T} = \{l^n; n \in \mathbf{W}, l > 1\}$, where $l = \alpha^T$, $\alpha > 1$

$$V^k(l^n) = V^k(\alpha^{nT}) := X(\alpha^{nT+k}). \tag{4.1}$$

This is by the fact that $V^k(l^n)$ are DT-SS process for $k = 0, 1, \ldots, T-1$ according to the definition 2.5 for $s = \alpha^k$ and all $n, n_1, n_2 \in \mathbf{W}$

$$\mathrm{Cov}\big(V^j(l^{n+n_1}), V^i(l^{n+n_2})\big) = \mathrm{Cov}\big(X(\alpha^{(n+n_1)T+j}), X(\alpha^{(n+n_2)T+i})\big)$$

$$= \alpha^{2nTH}\mathrm{Cov}\big(X(\alpha^{n_1T+j}), X(\alpha^{n_2T+i})\big) = l^{2nH}\mathrm{Cov}\big(V^j(l^{n_1}), V^i(l^{n_2})\big).$$

This equality provide the assertion (b) of definition 4.1 with $s_i = s_j = 1$, $i, j = 0, \cdots, T-1$. Let $Q^H(n, \tau) = [Q^H_{jk}(n, \tau)]_{j,k=0,1,\ldots,T-1}$ be the covariance matrix of $V(l^n)$, then for $j, k = 0, 1, \ldots, T-1$

$$Q^H_{jk}(n, \tau) = E[V^j(l^{n+\tau})V^k(l^n)] = E[X(\alpha^{(n+\tau)T+j})X(\alpha^{nT+k})], \qquad \tau \in \mathbf{W}.$$

Theorem 4.1 *Let* $\{X(\alpha^n), n \in \mathbf{W}\}$ *be a DT-SIM process with the covariance function* $R^H_n(\tau)$ *and let* $\{V(l^n), n \in \mathbf{W}\}$, *defined in* (4.1), *be its associated T-dimensional discrete time self-similar process with covariance function* $Q^H(n, \tau)$. *Then*

$$Q^H(n, \tau) = \alpha^{2nHT}Q^H(\tau) = \alpha^{2nHT}C_H R_H[\tilde{h}(\alpha^{T-1})]^\tau, \qquad \tau \in \mathbf{W} \tag{4.2}$$

where $\tilde{h}(\cdot)$ is defined by (3.3) and the matrices C_H and R_H are given by

$$C_H = \begin{bmatrix} C_{00}^H & C_{01}^H & \cdots & C_{0,T-1}^H \\ C_{10}^H & C_{11}^H & \cdots & C_{1,T-1}^H \\ \vdots & \vdots & \vdots & \vdots \\ C_{T-1,0}^H & C_{T-1,1}^H & \cdots & C_{T-1,T-1}^H \end{bmatrix},$$

$C_{jk}^H = \tilde{h}(\alpha^{j-1})[\tilde{h}(\alpha^{k-1})]^{-1}$, and

$$R_H = \begin{bmatrix} R_0^H(0) & 0 & \cdots & 0 \\ 0 & R_1^H(0) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & R_{T-1}^H(0) \end{bmatrix}.$$

Proof: As $X(\cdot)$ is DT-SI with parameter space $\{\alpha^k, k \in \mathbf{W}\}$ and scale α^T, so by (3.1)

$$Q_{jk}^H(n,\tau) = \alpha^{2nHT} E[X(\alpha^{\tau T+j})X(\alpha^k)] = \alpha^{2nHT} R_k^H(\tau T + j - k). \tag{4.3}$$

Let $R_k^H(\tau T + j - k) = Q_{jk}^H(\tau)$, then

$$Q_{jk}^H(n,\tau) = \alpha^{2nHT} Q_{jk}^H(\tau)$$

hence

$$Q^H(\tau) = \begin{bmatrix} R_0^H(\tau T) & R_1^H(\tau T - 1) & \cdots & R_{T-1}^H(\tau T - (T-1)) \\ R_0^H(\tau T + 1) & R_1^H(\tau T) & \cdots & R_{T-1}^H(\tau T - (T-2)) \\ \vdots & \vdots & \vdots & \vdots \\ R_0^H(\tau T + T - 1) & R_1^H(\tau T + T - 2) & \cdots & R_{T-1}^H(\tau T) \end{bmatrix}$$

and by the Markov property of $X(\cdot)$ from theorem 3.1, for $0 \leqslant j - k \leqslant T - 1$

$$R_k^H(\tau T + j - k) = [\tilde{h}(\alpha^{T-1})]^\tau \tilde{h}(\alpha^{j-1})[\tilde{h}(\alpha^{k-1})]^{-1} R_k^H(0).$$

Let $C_{jk}^H = \tilde{h}(\alpha^{j-1})[\tilde{h}(\alpha^{k-1})]^{-1}$, so

$$Q_{jk}^H(\tau) = [\tilde{h}(\alpha^{T-1})]^\tau C_{jk}^H R_k^H(0), \qquad \tau \in \mathbf{W}. \tag{4.4}$$

Thus we can represent $Q^H(n,\tau)$ as

$$\alpha^{2nHT}[\tilde{h}(\alpha^{T-1})]^\tau \begin{bmatrix} C_{00}^H R_0^H(0) & C_{01}^H R_1^H(0) & \cdots & C_{0,T-1}^H R_{T-1}^H(0) \\ C_{10}^H R_0^H(0) & C_{11}^H R_1^H(0) & \cdots & C_{1,T-1}^H R_{T-1}^H(0) \\ \vdots & \vdots & \vdots & \vdots \\ C_{T-1,0}^H R_0^H(0) & C_{T-1,1}^H R_1^H(0) & \cdots & C_{T-1,T-1}^H R_{T-1}^H(0) \end{bmatrix}. \square$$

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